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tion the enthalpy of the body changes in value because the form of the function changes from the form (2) to the form (3). Suppressing the constants $p$, $\theta$, $M_k$, this change of value is

$$G(M_j + \delta M_j) - G(M_j) - g_j\delta M_j.$$

Since this expression denotes the heat absorbed by the body $(M_j, M_k, \delta M_j)$ during the dilution, we have that the heat absorbed per unit mass of diluent added is the limit

$$\lim_{\delta M_j \to 0} \frac{G(M_j + \delta M_j) - G(M_j) - g_j\delta M_j}{\delta M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

This quantity is the "heat of dilution" $\Delta_j$ of the solution $(M_1, M_2)$ for dilution by the $j$th component of the mixture.

Now by differentiating (1) we find

$$\frac{\partial \Delta G}{\partial M_j} = \frac{\partial G}{\partial M_j} - g_j.$$

Hence $\Delta_j$ is equal to the derivative $\partial\Delta G/\partial M_j$ of the heat of mixing. Further, since it is known that $G(p, \theta, M_1, M_2)$ is homogeneous of degree one in $M_1$, $M_2$, and hence by (1) that $\Delta G$ is homogeneous of degree one in these variables, we have

$$\Delta G = M_1\Delta_1 + M_2\Delta_2.$$

It thus appears that the heat of mixing of a solution is a linear function of the two heats of dilution.

## II. DETERMINANTS IN ELEMENTARY ANALYTIC GEOMETRY.

By F. A. FORAKER, University of Pittsburgh.

The general purpose of this paper is to indicate how some results in elementary analytic geometry may be conveniently expressed in determinant form. We shall consistently use the notation

$$(1) \qquad \Delta = \begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}, \qquad D = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix};$$

and shall in the usual way indicate the cofactors of elements of determinants by corresponding capital letters. The points $P_1$, $P_2$, $P_3$ will be understood to have coördinates $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ respectively. It is well known that the area of the triangle $P_1P_2P_3$ is $\frac{1}{2}\Delta$, and that therefore the points $P_1$, $P_2$, $P_3$ are collinear if and only if $\Delta = 0$.

The equation of the line joining $P_1$ and $P_2$ is

$$(2) \qquad \begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0;$$

in particular, the equation of a line having intercepts $a$, $b$ is

$$\begin{vmatrix} x, & y, & 1 \\ a, & 0, & 1 \\ 0, & b, & 1 \end{vmatrix} = 0. \tag{3}$$

The equations of lines respectively parallel and perpendicular to (2) through $P_3$ are

$$\begin{vmatrix} x - x_3, & y - y_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0 \tag{4}$$

and

$$\begin{vmatrix} y_3 - y, & x - x_3, & 0 \\ x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \end{vmatrix} = 0. \tag{5}$$

The latter equation may also be written:

$$\begin{vmatrix} x, & y, & 1 \\ x_3, & y_3, & 1 \\ X_3, & Y_3, & 0 \end{vmatrix} = 0. \tag{6}$$

The equation of a line through $P_1$ with slope $m$ is

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ 1, & m, & 0 \end{vmatrix} = 0; \tag{7}$$

the normal form of the equation of a line may be written

$$\begin{vmatrix} x, & y, & 1 \\ p, & 0, & \cos\alpha \\ 0, & p, & \sin\alpha \end{vmatrix} = 0, \tag{8}$$

provided $p \neq 0$.

All these equations are of the general type

$$\begin{vmatrix} x, & y, & 1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0, \tag{9}$$

or may be readily reduced to that type. Conversely, (9) represents a straight line unless $A_1 = B_1 = 0$. From (9) any of the usual forms may be derived; it is seen at once that the intercepts and the slope are

$$a = -C_1/A_1, \qquad b = -C_1/B_1; \qquad m = -A_1/B_1.$$

The equation of a circle through the points $P_1$, $P_2$, $P_3$ is

$$\begin{vmatrix} x^2 + y^2, & x, & y, & 1 \\ x_1^2 + y_1^2, & x_1, & y_1, & 1 \\ x_2^2 + y_2^2, & x_2, & y_2, & 1 \\ x_3^2 + y_3^2, & x_3, & y_3, & 1 \end{vmatrix} = 0; \tag{10}$$

this may be written

$$\Delta(x^2+y^2) - \left| x_i^2+y_i^2,\ y_i,\ 1 \right| x - \left| x_i,\ x_i^2+y_i^2,\ 1 \right| y = \left| x_i^2+y_i^2,\ x_i,\ y_i \right|,$$

where the meaning of the abbreviated determinants is obvious. Thus the circumcenter of the triangle $P_1P_2P_3$ has the coördinates

$$C_x = \frac{1}{2\Delta}\left| x_i^2+y_i^2,\quad y_i,\quad 1 \right|, \qquad C_y = \frac{1}{2\Delta}\left| x_i,\quad x_i^2+y_i^2,\quad 1 \right|. \tag{11}$$

The median through the vertex $P_1$ of the triangle $P_1P_2P_3$ has the equation

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_2+x_3, & y_2+y_3, & 2 \end{vmatrix} = 0, \tag{12}$$

which may be written in the form

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ x_1+x_2+x_3, & y_1+y_2+y_3, & 3 \end{vmatrix} = 0.$$

From symmetry it is seen at once that the three medians intersect in a single point, the centroid, having the coördinates

$$G_x = \tfrac{1}{3}(x_1+x_2+x_3), \qquad G_y = \tfrac{1}{3}(y_1+y_2+y_3). \tag{13}$$

The altitudes of the triangle $P_1P_2P_3$ through $P_1$ and $P_2$ have respectively the equations

$$\begin{vmatrix} x, & y, & 1 \\ x_1, & y_1, & 1 \\ y_2-y_3, & x_3-x_2, & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} x, & y, & 1 \\ x_2, & y_2, & 1 \\ y_1-y_3, & x_3-x_1, & 0 \end{vmatrix} = 0.$$

Solving, we find

$$x = \frac{\begin{vmatrix} x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_3-y_2 \\ x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_3-y_1 \end{vmatrix}}{\begin{vmatrix} x_3-x_2, & y_3-y_2 \\ x_3-x_1, & y_3-y_1 \end{vmatrix}}.$$

If we interchange rows, change the signs in certain columns, and border with the quantities indicated, we have

$$x = \frac{\begin{vmatrix} x_2x_3-x_1x_2+y_2y_3-y_1y_2, & y_1-y_3, & 0 \\ x_1x_3-x_1x_2+y_1y_3-y_1y_2, & y_2-y_3, & 0 \\ x_1x_2+y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1-x_3, & y_1-y_3, & 0 \\ x_2-x_3, & y_2-y_3, & 0 \\ x_3, & y_3, & 1 \end{vmatrix}};$$

so that finally

$$x = \frac{\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix}}{-\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}$$

In like manner we may obtain $y$. The results involve the three subscripts symmetrically; therefore the point must be the intersection of each pair of altitudes. As the common intersection of the three altitudes, the orthocenter, we thus have

$$(14) \qquad \begin{aligned} O_x &= -\frac{1}{\Delta}\begin{vmatrix} x_2x_3 + y_2y_3, & y_1, & 1 \\ x_3x_1 + y_3y_1, & y_2, & 1 \\ x_1x_2 + y_1y_2, & y_3, & 1 \end{vmatrix}, \\ O_y &= -\frac{1}{\Delta}\begin{vmatrix} x_1, & x_2x_3 + y_2y_3, & 1 \\ x_2, & x_3x_1 + y_3y_1, & 1 \\ x_3, & x_1x_2 + y_1y_2, & 1 \end{vmatrix}. \end{aligned}$$

From (11), (13), (14) we have

$$S = \Delta(3G_x - 2C_x - O_x) = \begin{vmatrix} r_1 & y_1 & 1 \\ r_2 & y_2 & 1 \\ r_3 & y_3 & 1 \end{vmatrix},$$

where

$$\begin{aligned} r_1 &= x_1(x_1 + x_2 + x_3) - (x_1^2 + y_1^2) + (x_2x_3 + y_2y_3) \\ &= (x_2x_3 + x_3x_1 + x_1x_2) + (y_2y_3 - y_1^2), \end{aligned}$$

with similar expressions for $r_2$, $r_3$. Hence

$$S = (x_2x_3 + x_3x_1 + x_1x_2)\begin{vmatrix} 1, & y_1, & 1 \\ 1, & y_2, & 1 \\ 1, & y_3, & 1 \end{vmatrix} + \begin{vmatrix} y_2y_3, & y_1, & 1 \\ y_3y_1, & y_2, & 1 \\ y_1y_2, & y_3, & 1 \end{vmatrix} - \begin{vmatrix} y_1^2, & y_1, & 1 \\ y_2^2, & y_2, & 1 \\ y_3^2, & y_3, & 1 \end{vmatrix}.$$

The first determinant vanishes; each of the others is equal to $(y_3 - y_2)(y_1 - y_3)(y_2 - y_1)$; therefore $S = 0$. We may similarly examine the $y$-coördinates; we find that

$$\begin{aligned} 3G_x - 2C_x - O_x &= 0, \\ 3G_y - 2C_y - O_y &= 0. \end{aligned}$$

Thus we have proved the theorem: *The centroid lies one-third of the distance from the circumcenter to the orthocenter, along the line joining them.*

If we have the equations of three lines given in the form

$$(15) \qquad a_ix + b_iy + c_i = 0 \qquad [i = 1, 2, 3],$$

the lines will form a triangle if $D$, $C_1$, $C_2$, $C_3$ are all different from zero. The vertices of the triangle are then

$$
\begin{aligned}
x_{23} &= A_1/C_1, & y_{23} &= B_1/C_1. \\
x_{31} &= A_2/C_2, & y_{31} &= B_2/C_2. \\
x_{12} &= A_3/C_3, & y_{12} &= B_3/C_3.
\end{aligned} \tag{16}
$$

The coördinates of the circumcenter are seen by (11) and (16) to be[1]

$$
\begin{aligned}
C_x &= \frac{1}{2D^2}\left|\frac{A_i^2+B_i^2}{C_i}, \quad B_i, \quad C_i\right|, \\
C_y &= \frac{1}{2D^2}\left|A_i, \quad \frac{A_i^2+B_i^2}{C_i}, \quad C_i\right|.
\end{aligned} \tag{17}
$$

The coördinates of the centroid are

$$
C_x = \tfrac{1}{3}\left(\frac{A_1}{C_1}+\frac{A_2}{C_2}+\frac{A_3}{C_3}\right), \qquad C_y = \tfrac{1}{3}\left(\frac{B_1}{C_1}+\frac{B_2}{C_2}+\frac{B_3}{C_3}\right). \tag{18}
$$

The coördinates of the orthocenter are

$$
\begin{aligned}
O_x &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} (A_2A_3+B_2B_3)C_1^2, & B_1, & C_1 \\ (A_3A_1+B_3B_1)C_2^2, & B_2, & C_2 \\ (A_1A_2+B_1B_2)C_3^2, & B_3, & C_3 \end{vmatrix}, \\
O_y &= -\frac{1}{C_1C_2C_3D^2}\begin{vmatrix} A_1, & (A_2A_3+B_2B_3)C_1^2, & C_1 \\ A_2, & (A_3A_1+B_3B_1)C_2^2, & C_2 \\ A_3, & (A_1A_2+B_1B_2)C_3^2, & C_3 \end{vmatrix}.
\end{aligned} \tag{19}
$$

The area of the triangle is $D^2/(2C_1C_2C_3)$.

A few of the above results are well known; the others are easily proved. This paper is intended only as an introduction to work along this line and is not supposed to be exhaustive in any sense.

## III. THE HISTORY OF MATHEMATICS IN ELEMENTARY INSTRUCTION.[2]

By LAURIN ZILLIACUS, Bedales School, Petersfield, England.

The General Education Board has recently announced the appropriation of $16,000 to be used by the National Committee on Mathematical Requirements in financing a study looking to improvements in the mathematical curriculum of the secondary schools of America. If there is to be any recasting of curricula, I earnestly hope that a full use will be made of the history of mathematics. A short perusal of this history, as given in popular works such as Ball's *History of Mathematics*, would, I feel sure, cause a majority of present-day teachers of mathematics to modify considerably both the outlines of their courses and their

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[1] Since $|A_1B_2C_3| = |a_1b_2c_3|^2$.

[2] Extract from a letter to Professor H. W. Tyler.